CARNEGIE LEARNING

# 1 Reasoning with Shapes

**Pacing: 39 Days**

## Topic 1: Composing and Decomposing Shapes

Students investigate and conjecture about geometric figures. They use circles and their defining characteristics as the template upon which to construct lines, angles, triangles, and quadrilaterals. Students use reasoning to conjecture about the relationships they notice, preparing them for formal proof in future topics.

**Standards:** G.CO.9†, G.CO.10†, G.CO.11†, G.C.1†, G.C.3† **Pacing:** 10 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **Running Circles Around Geometry**<br>Using Circles to Make Conjectures | G.CO.9†<br>G.CO.10†<br>G.C.1† | 2 | Circles are used to make conjectures about line and angle relationships. Students construct a circle, a perpendicular bisector of a diameter, and a chord to identify circle parts. Next, they conjecture about angle relationships given parallel lines intersected by a transversal. Students make conjectures related to inscribed angles and angles formed at the point of tangency when two lines intersect at a point outside the circle. | • When you conjecture, you use what you know through experience and reasoning to presume that something is true. The statement of a conjecture, once proven, is then called a theorem.<br>• Circles can be helpful in constructing geometric figures in order to make conjectures about line and angle relationships. |
| 2 | **The Quad Squad**<br>Conjectures About Quadrilaterals | G.CO.11†<br>G.C.3 | 2 | Students use circles to investigate and conjecture about the properties of quadrilaterals. Students construct several quadrilaterals from the diameters of concentric circles. Using the measurements of sides and angles, they are able to name the quadrilaterals. Students make conjectures about the diagonals and angle relationships of kites and isosceles trapezoids. They identify quadrilaterals with given properties and then describe how to construct various quadrilaterals given only one diagonal. Students conjecture about the figure formed by adjacent midsegments of quadrilaterals and the measure of the midsegment of a trapezoid in relation to its bases. Finally, they conjecture about the sum of the measures of opposite angles of different cyclic quadrilaterals. | • The diagonals of any convex quadrilateral create two pairs of vertical angles and four linear pairs of angles.<br>• Parallelograms, rhombi, and kites have diagonals that are not congruent.<br>• Rectangles, squares, and isosceles trapezoids have congruent diagonals.<br>• Circles can be helpful in understanding that the diagonals of parallelograms bisect each other, the diagonals of rectangles are congruent, and the diagonals of kites are perpendicular.<br>• The measure and relationship of the diagonals of quadrilaterals can be used to make conjectures about quadrilaterals.<br>• The relationship of the interior angles of quadrilaterals can be used to make conjectures about quadrilaterals.<br>• The midsegment of a quadrilateral is any line segment that connects two midpoints of the sides of the quadrilateral.<br>• A quadrilateral whose vertices all lie on a single circle is a cyclic quadrilateral. |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

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| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **Tri- Tri- Tri- and Separate Them**<br>Conjectures About Triangles | G.CO.10† | 2 | Students decompose quadrilaterals to form the triangles they investigate in this lesson. They write the converses of conditional statements and then explore the converse of their base angles conjecture for isosceles triangles. Students construct an equilateral triangle and conjecture about the sum of the interior and exterior angle measures of a triangle. Students use a circle diagram to make conjectures about triangle inequality and triangle midsegments. | • Circles can be helpful in constructing geometric figures to make conjectures about triangles.<br>• A convex quadrilateral can be divided by any one of its diagonals into two triangles.<br>• The converse of a statement is different from the original statement and is formed by interchanging the hypothesis and conclusion of the original statement.<br>• The truth value of a conditional statement and its converse are not necessarily the same.<br>• The base angles of an isosceles triangle are congruent.<br>• A point that lies on a perpendicular bisector of a line segment is equidistant from the endpoints of the line segment.<br>• The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.<br>• The sum of the measures of the interior angles of a triangle is 180°.<br>• The length of the third side of a triangle cannot be equal to or greater than the sum of the measures of the other two sides.<br>• The midsegment of a triangle is one-half the measure and parallel to the third side.<br>• A conjecture is a statement believed to be true based on observations. A conjecture must be proved with definitions and theorems to be fully accepted. |
| 4 | **What's the Point?**<br>Points of Concurrency | G.CO.10†<br>G.C.3 | 2 | Students construct the four points of concurrency—the incenter, circumcenter, centroid, and orthocenter. They construct perpendicular bisectors, angle bisectors, medians, and altitudes to locate these points in acute, obtuse, right, and equilateral triangles. They use the circumcenter to circumscribe a circle about a triangle and the incenter to inscribe a circle in a triangle. Students use their constructions to make conjectures. | • A point of concurrency is a point at which three lines, rays, or line segments intersect.<br>• The circumcenter is the point of concurrency of the three perpendicular bisectors of the sides of a triangle, and it is equidistant from each vertex of the triangle.<br>• The circumcenter can be used to circumscribe a circle about a triangle.<br>• The incenter is the point at which the three angle bisectors of a triangle are concurrent and it is equidistant from each side of the triangle.<br>• The incenter can be used to construct a circle inscribed in a triangle.<br>• The median of a triangle is a line segment formed by connecting a vertex of a triangle to the midpoint of the opposite side of the triangle.<br>• The centroid is the point at which the three medians of a triangle are concurrent.<br>• The distance from the centroid to the vertex is twice the distance from the centroid to the midpoint of the opposite side.<br>• The orthocenter is the point at which the three altitudes of a triangle are concurrent. |
| **Learning Individually with MATHia or Skills Practice** | | | 2 | Students practice identifying geometric entities from their names, writing names for various geometric entities, and identifying when an entity has multiple possible names. They then practice writing measure statements for segments and angles using appropriate notation. Students calculate the measure of an arc or an angle using the definition of a central angle, the Arc Addition Postulate, or the Inscribed Angle Theorem. They are shown an inscribed quadrilateral and prove the Inscribed Quadrilateral-Opposite Angles Conjecture. Students then use the theorem to determine the measure of an angle in an inscribed quadrilateral given the measure of the opposite angle. | |

*Pacing listed in 45-minute days
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## Topic 2: Justifying Line and Angle Relationships

Students are introduced to formal geometric reasoning. They learn how to write formal proofs—flow chart, two-column and paragraph proofs, in addition to proof by construction and the algebraic proofs that they used previously—and then prove many of the conjectures that they made in the previous topic. Students begin by proving foundational theorems and then prove theorems related to angle pairs formed when parallel lines are intersected by a transversal. They prove conjectures about the angles on the interior and exterior of polygons and then focus on conjectures about the relationships between sides and angles in triangles. Finally, students prove theorems about angle relationships formed by chords and lines inside and outside of circles.

**Standards:** N.RN.2, G.CO.9, G.CO.10, G.CO.11, G.C.2, G.C.3, G.C.4(+) **Pacing:** 16 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **Proof Positive**<br>Forms of Proof | G.CO.9 | 3 | Students apply real number properties to angle measures, line segments, and distances. Next, they informally use constructions to reason about a conditional statement. Students are introduced to flowchart and two-column proofs and analyze worked examples of both forms to prove the same statement. They then complete partial proofs to prove the Right Angle Congruence Theorem and the Congruent Supplement Theorem. Students analyze a flowchart proof of the Vertical Angle Theorem before writing a two-column proof for the same theorem using the Congruent Supplement Theorem. They use these theorems to determine unknown angle measures. Finally, students are introduced to paragraph proofs and demonstrate what they have learned using complete sentences. | • The Addition Property of Equality, the Subtraction Property of Equality, the Reflexive Property, the Substitution Property, and the Transitive Property can be applied to angle measures, segment measures, and distances.<br>• A construction proof, two-column proof, flow chart proof, and paragraph proof are all acceptable forms of reasoning about geometric relationships.<br>• The Right Angle Congruence Postulate states: "All right angles are congruent."<br>• The Congruent Supplement Theorem states: "If two angles are supplements of the same angle or of congruent angles, then the angles are congruent."<br>• The Vertical Angle Theorem states: "Vertical angles are congruent." |

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| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 2 | **A Parallel Universe**<br>Proving Parallel Line Theorems | G.CO.9 | 2 | Students explore theorems related to parallel lines cut by a transversal, proving both that special angle pairs are congruent given parallel lines and the converse statements—that two lines are parallel given the congruence of special angle pairs. Students begin by proving the Corresponding Angles Theorem using what they know about translations, and they prove the remaining theorems in flowcharts and two-column format using definitions, postulates, and already proven theorems. Students continue to investigate the process of creating proofs, building proof plans to help them connect if/then statements using deductive reasoning. | • The Corresponding Angle Theorem states: "If two parallel lines are cut by a transversal, then corresponding angles are congruent."<br>• The Corresponding Angle Converse Theorem states: "If two lines cut by a transversal form congruent corresponding angles, then the lines are parallel."<br>• The Same-Side Interior Angle Theorem states: "If two parallel lines are cut by a transversal, then same-side interior angles are supplementary."<br>• The Same-Side Interior Angle Converse Theorem states: "If two lines cut by a transversal form supplementary same-side interior angles, then the lines are parallel."<br>• The Alternate Interior Angle Theorem states: "If two parallel lines are cut by a transversal, then alternate interior angles are congruent."<br>• The Alternate Interior Angle Converse Theorem states: "If two lines cut by a transversal form congruent alternate interior angles, then the lines are parallel."<br>• The Same-Side Exterior Angle Theorem states: "If two parallel lines are cut by a transversal, then same-side exterior angles are supplementary."<br>• The Same-Side Exterior Angle Converse Theorem states: "If two lines cut by a transversal form supplementary same-side exterior angles, then the lines are parallel."<br>• The Alternate Exterior Angle Theorem states: "If two parallel lines are cut by a transversal, then alternate exterior angles are congruent."<br>• The Alternate Exterior Angle Converse Theorem states: "If two lines cut by a transversal form congruent alternate exterior angles, then the lines are parallel."<br>• The Perpendicular/Parallel Line Theorem states: "If two lines are perpendicular to the same line, then the two lines are parallel to each other." |
| 3 | **Ins and Outs**<br>Interior and Exterior Angles of Polygons | G.CO.10 | 2 | Students investigate the Triangle Sum Theorem and then prove the theorem using what they know about congruent angle pairs formed from parallel lines and a transversal. Students then explain how the Exterior Angle Theorem can be demonstrated using the same diagram as the one used to prove the Triangle Sum Theorem. Students generalize this activity to derive a formula that can be used to determine the sum of the interior angle measures of any polygon and also determine the sum of the exterior angle measures of any polygon. Finally, students demonstrate what they have learned in the lesson by solving a variety of mathematical problems. | • The Triangle Sum Theorem states: "The sum of the measures of the interior angles of a triangle is equal to 180°."<br>• The Exterior Angle Theorem states: "The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles."<br>• The sum of the measures of the interior angles of a quadrilateral is equal to 360°.<br>• For a polygon with n sides, the sum of its interior angle measures is equal to $180(n - 2)°$.<br>• For a regular polygon with n sides, the measure of each interior angle is equal to $180(n - 2)°/n$.<br>• For a polygon with n sides, the sum of the measures of the exterior angles is equal to 360°. |

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| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 4 | **Identical Twins**<br>Perpendicular Bisector and Isosceles Triangle Theorems | N.RN.2<br>G.CO.9<br>G.CO.10 | 2 | Students use their knowledge of Side-Angle- Side (SAS), Side-Side-Side (SSS), or Angle- Side-Angle (ASA) Theorems to explain why pairs of triangles are congruent. The term CPCTC (corresponding parts of congruent triangles are congruent) is defined as a reason that can be used after two triangles are proved congruent. Students then investigate and prove the Perpendicular Bisector Theorem using CPCTC and analyze a worked example of its converse. They then use CPCTC to prove the Isosceles Triangle Base Angles Theorem and its converse. Students use the converse of the Perpendicular Bisector Theorem to demonstrate the 30°-60°-90° Triangle Theorem using algebra. They then use the Isosceles Triangle Base Angles Theorem to algebraically demonstrate the 45°-45°-90° Triangle Theorem. Finally, students reason about the Hypotenuse-Angle Theorem and the Angle-Angle-Side Congruence Theorem and solve a variety of mathematical and real-world problems using what they learned in the lesson. | • The Perpendicular Bisector Theorem states: "The points on a perpendicular bisector of a line segment are equidistant from the segment's endpoints."<br>• The Perpendicular Bisector Converse Theorem states: "If a point is equidistant from the endpoints of a line segment, then the point lies on the perpendicular bisector of the line segment."<br>• The Isosceles Triangle Base Angles Theorem states: "If two sides of a triangle are congruent, then the angles opposite these sides are congruent."<br>• The Isosceles Triangle Base Angles Converse Theorem states: "If two angles of a triangle are congruent, then the sides opposite these angles are congruent."<br>• The 30°-60°-90° Triangle Theorem states: "The length of the hypotenuse in a 30°-60°-90° triangle is 2 times the length of the shorter leg, and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg."<br>• The 45°-45°-90° Triangle Theorem states: "The length of the hypotenuse in a 45°-45°-90° triangle is $\sqrt{2}$ times the length of a leg.<br>• The Hypotenuse-Angle (HA) Congruence Theorem states: "If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and an acute angle of another right triangle, then the two triangles are congruent."<br>• The Angle-Angle-Side (AAS) Congruence Theorem states: "If two angles and the non-included side of one triangle are congruent to two angles and the non-included side of another triangle, then the two triangles are congruent." |

*Pacing listed in 45-minute days

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| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 5 | **Corners in a Round Room**<br><br>Angle Relationships Inside and Outside Circles | G.C.2<br>G.C.3<br>G.C.4(+) | 2 | Students reason about arc measures associated with a clockface and conclude that the measures of two central angles of the same circle (or congruent circles) have corresponding congruent minor arcs. They use a two-column proof to prove one case of the Inscribed Angle Theorem and algebraic reasoning to prove the other two cases. Students then prove two theorems associated with inscribed polygons using the Inscribed Angle Theorem. Next, they explore and prove theorems for determining the measures of angles located on the inside and outside of a circle. They construct a tangent line to a circle from a point outside the circle. A proof by contradiction is provided to show a perpendicular relationship exists when the radius of a circle is drawn to a point of tangency. Finally, students use the theorems they have proven to determine the measures of arcs and angles of a circle. | • The Arc Addition Postulate states: "The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs."<br>• The measure of a central angle is equal to the measure of its intercepted arc.<br>• The Inscribed Angle Theorem states: "The measure of an inscribed angle is equal to half the measure of its intercepted arc."<br>• The Inscribed Right Triangle–Diameter Theorem states: "When a triangle is inscribed in a circle such that one side of the triangle is a diameter, the triangle is a right triangle."<br>• The Inscribed Quadrilateral–Opposite Angles Theorem states: "When a quadrilateral is inscribed in a circle, the opposite angles are supplementary."<br>• The Interior Angles of a Circle Theorem states: "If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the interior of the circle, then the measure of the angle is half of the sum of the measures of the arcs intercepted by the angle and its vertical angle."<br>• The Exterior Angles of a Circle Theorem states: "If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the exterior of the circle, then the measure of the angle is half of the difference of the measures of the arcs intercepted by the angle."<br>• The Tangent to a Circle Theorem states: "A line drawn tangent to a circle is perpendicular to a radius of the circle drawn to the point of tangency." |
| **Learning Individually with MATHia or Skills Practice** | | | 5 | Students identify given angle measures and justify their reasoning. They learn to write flowchart proofs and then convert them to two-column proofs. They calculate the measures of angles and sides in polygons and along parallel lines before writing formal proofs of the known relationships. | |

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## Topic 3: Using Congruence Theorems

Students use the theorems that they proved in *Justifying Line and Angle Relationships* to prove additional theorems. First, they prove congruence theorems specific to right triangles. Then, students prove properties of quadrilaterals. Finally, they prove theorems about relationships between chords of congruent circles.

**Standards:** G.CO.10, G.CO.11, G.C.2 **Pacing:** 13 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **SSS, SAS, AAS, … S.O.S!**<br><br>Using Triangle Congruence to Determine Relationships Between Segments | G.CO.10<br>G.SRT.5 | 2 | Students construct a right triangle in a circle given a leg length and a hypotenuse length. They compare their constructions and notice that the triangles are congruent to each other. Students then prove this Hypotenuse-Leg Congruence Theorem using both a two-column proof and algebraic reasoning. Students use reasoning to demonstrate that the Leg-Leg Congruence Theorem and the Leg-Angle Congruence Theorem must be true, given the triangle congruence theorems they know. They then apply these congruence theorems to solve problems. Finally, students analyze a proof of the Tangent Segment Theorem and use it to solve additional problems. | • The Hypotenuse-Leg Congruence Theorem states: "If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent."<br>• The Leg-Leg Congruence Theorem states: "If the two corresponding shorter legs of two right triangles are congruent, then the triangles are congruent."<br>• The Leg-Angle Congruence Theorem states: "If the leg and an acute angle of one right triangle are congruent to the corresponding leg and acute angle of another triangle, then the triangles are congruent."<br>• The Leg-Leg and Leg-Angle Congruence Theorems can be justified using SSS, SAS, ASA, and/or AAS triangle congruence.<br>• The Tangent Segment Theorem states: "If two tangent segments are drawn from the same point on the exterior of a circle, then the tangent segments are congruent." |
| 2 | **Props to You**<br><br>Properties of Quadrilaterals | G.CO.11 | 2 | Students verify the properties of a parallelogram, a rhombus, a rectangle, a square, a trapezoid, and a kite using formal two-column or paragraph formats, as well as informal reasoning. Students apply the theorems to arrive at solutions to problem situations. | • A parallelogram is defined as a quadrilateral with opposite sides parallel. The properties of a parallelogram include:<br>  • The opposite sides of a parallelogram are congruent.<br>  • The opposite angles of a parallelogram are congruent.<br>  • The diagonals of a parallelogram bisect each other.<br>  • If one pair of opposite sides of a quadrilateral are both parallel and congruent, then the quadrilateral is a parallelogram.<br>• A rhombus is defined as a parallelogram with all sides congruent. The properties of a rhombus include:<br>  • The diagonals of a rhombus bisect the vertex angles.<br>  • The diagonals of a rhombus are perpendicular to each other.<br>• A rectangle is defined as a parallelogram with all angles congruent. The diagonals of a rectangle are congruent.<br>• A square is defined as a parallelogram with all angles congruent and all sides congruent. The square has all of the properties of the rectangle and rhombus.<br>• An isosceles trapezoid is defined as a trapezoid with congruent non-parallel sides. The base angles of an isosceles trapezoid are congruent.<br>• The Trapezoid Midsegment Theorem states: "The midsegment of a trapezoid is parallel to each of the bases and its length is one half the sum of the lengths of the bases."<br>• A kite is defined as a quadrilateral with two pairs of consecutive congruent sides. The properties of a kite include:<br>  • One diagonal of a kite is a line of symmetry.<br>  • One diagonal of a kite bisects a pair of opposite angles.<br>  • One diagonal of a kite is the perpendicular bisector of the other diagonal. |

*Pacing listed in 45-minute days

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| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **Three-Chord Song**<br>Relationships Between Chords | G.C.2 | 1 | Students explore a problem situation which asks them to determine the diameter of a circular plate given only a broken piece of the plate. Students conjecture about methods that may be used to determine the diameter. They then prove the Diameter-Chord Theorem, Equidistant Chord Theorem, and Equidistant Chord Converse Theorem. They also prove the Congruent Chord–Congruent Arc Theorem and its converse. Finally, students revisit and solve the broken plate problem from the Getting Started. | • The Diameter–Chord Theorem states: "If a circle's diameter is perpendicular to a chord, then the diameter bisects the chord and<br>• The Equidistant Chord Theorem states: "If two chords of the same circle or congruent circles are congruent, then they are equidistant from the center of the circle."<br>• The Equidistant Chord Converse Theorem states: "If two chords of the same circle or congruent circles are equidistant from the center of the circle, then the chords are congruent."<br>• The Congruent Chord–Congruent Arc Theorem states: "If two chords of the same circle or congruent circles are congruent, then their corresponding arcs are congruent."<br>• The Congruent Chord–Congruent Arc Converse Theorem states: "If two arcs of the same circle or congruent circles are congruent, then their corresponding chords are congruent." |
| **Learning Individually with MATHia or Skills Practice** | | G.CO.7<br>G.CO.8<br>G.CO.10<br>G.CO.11 | 8 | Students use the proof tool to prove triangles congruent using SSS, SAS, AAS, ASA, HL, and HA. They use CPCTC to prove other theorems related to angles and sides in triangles. They review the properties of parallelograms and use them to solve for unknown parts of quadrilaterals and parallelograms before proving that they are true in all cases. Finally, students investigate circles and calculate the measures of arcs, central angles, and inscribed angles. | |

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CARNEGIE LEARNING

# 2 Investigating Proportionally

**Pacing: 36 Days**

## Topic 1: Similarity

Students build on what they know about rigid motion transformations to formally define similarity transformations. They perform dilations on figures and describe the similarity transformations necessary to map one figure onto another. Students establish triangle similarity criteria and use them to both determine the similarity of triangles and to prove theorems about proportionality. Geometric mean is defined, and students use it to solve problems and to prove the Pythagorean Theorem using similarity. Finally, they use the concepts of similarity and proportionality to partition directed line segments into given ratios.

**Standards:** G.SRT.1, G.SRT.1a, G.SRT.1b, G.SRT.2, G.SRT.3, G.SRT.4, G.SRT.5, G.GPE.6 **Pacing:** 15 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **Big, Little, Big, Little**<br>Dilating Figures to Create Similar Figures | G.SRT.1b<br>G.SRT.2 | 1 | Students perform dilations on triangles and other figures both on and off of the coordinate plane. They explore the ratios formed as a result of dilation and recall scale factor. Similar triangles are defined and students explore the relationships between the corresponding sides and between the corresponding angles. Students then use similarity statements to draw similar triangles, and describe the similarity transformations necessary to map one triangle onto another as an alternate approach to showing similarity. | • A dilation is a transformation that enlarges, reduces, or keeps congruent a pre-image to create an image.<br>• The center of dilation is a fixed point at which a figure is dilated.<br>• The scale factor of a dilation is the ratio of the distance from the center of dilation to a point on the image to the distance from the center of dilation to the corresponding point on the pre-image.<br>• When the scale factor is greater than 1, the dilation is an enlargement. When the scale factor is between 0 and 1, the dilation is a reduction. When the scale factor is exactly 1, the dilation produces a congruent figure. |
| 2 | **Similar Triangles or Not?**<br>Establishing Triangle Similarity Criteria | G.SRT.2<br>G.SRT.3 | 2 | Students explore methods for proving triangles similar using construction tools and measuring tools. Then the Angle-Angle Similarity Theorem, Side-Side-Side Similarity Theorem, and Side-Angle-Side Similarity Theorem are stated, and students use these theorems to determine the similarity of triangles. The terms *included angle* and *included side* are defined. Some problem situations require the use of construction tools and others require measurement tools. | • The Angle-Angle Similarity Theorem states: "If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar."<br>• The Side-Side-Side Similarity Theorem states: "If all three corresponding sides of two triangles are proportional, then the triangles are similar."<br>• The Side-Angle-Side Similarity Theorem states: "If two of the corresponding sides of two triangles are proportional and the included angles are congruent, then the triangles are similar."<br>• An included angle is an angle formed by two consecutive sides of a figure.<br>• An included side is a line segment between two consecutive angles of a figure. |

*Pacing listed in 45-minute days

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| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **Keep It in Proportion**<br>Theorems About Proportionality | G.SRT.1a<br>G.SRT.4 | 4 | Students prove the Angle Bisector/Proportional Side Theorem, the Triangle Proportionality Theorem, the Proportional Segments Theorem, and the Triangle Midsegment Theorem. Students use these theorems to solve problems. | • The Angle Bisector/Proportional Side Theorem states: "A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as the lengths of the sides adjacent to the angle."<br>• The Triangle Proportionality Theorem states: "If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally."<br>• The Converse of the Triangle Proportionality Theorem states: "If a line divides the two sides of a triangle proportionally, then it is parallel to the third side."<br>• The Proportional Segments Theorem states: "If three parallel lines intersect two transversals, then they divide the transversals proportionally."<br>• The Triangle Midsegment Theorem states: "The midsegment of a triangle is parallel to the third side of the triangle and half the measure of the third side of the triangle."<br>• The medians of a triangle are concurrent. |
| 4 | **This Isn't Your Average Mean**<br>More Similar Triangles | G.SRT.3<br>G.SRT.4<br>G.SRT.5 | 1 | The term *geometric mean* is defined and is used in triangle theorems to solve for unknown measurements. Students practice using the Right Triangle/Altitude Similarity Theorem, the Right Triangle Altitude/Hypotenuse Theorem, and the Right Triangle Altitude/Leg Theorem to solve problems. Next, they are guided through the steps necessary to prove the Pythagorean Theorem using similar triangles. | • The Right Triangle/Altitude Similarity Theorem states: "If an altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other."<br>• The geometric mean of two numbers $a$ and $b$ is the number $x$ such that $a/x = x/b$.<br>• The Right Triangle Altitude/Hypotenuse Theorem states: "The measure of the altitude drawn from the vertex of the right angle of a right triangle to its hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse."<br>• The Right Triangle Altitude/Leg Theorem states: "If the altitude is drawn to the hypotenuse of a right triangle, each leg of the right triangle is the geometric mean between the hypotenuse and the segment of the hypotenuse adjacent to the leg."<br>• The Pythagorean Theorem states: "If $a$ and $b$ are the lengths of the legs of a right triangle and $c$ is the length of the hypotenuse, then $a^2 + b^2 = c^2$." |
| 5 | **Run It Up the Flagpole**<br>Application of Similar Triangles | G.SRT.5 | 2 | Indirect measurement takes students out of their classroom and school building to measure the height of objects such as flagpoles, tops of trees, telephone poles, or buildings using similar triangles. Additionally, students are given several situations in which they create proportions related to similar triangles to solve for unknown measurements. | • Indirect measurement is the process of using proportions related to similar triangles to determine a measurement when direct measurement is inconvenient or difficult.<br>• The mirror method and the shadow method are used to diagram similar triangles.<br>• The Law of Reflection states that the angle of incidence is congruent to the angle of reflection. |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 6 | **Jack's Spare Key**<br>Partitioning Segments in Given Ratios | G.GPE.6 | 1 | Students use the Midpoint Formula and Distance Formula to determine the midpoints of line segments on the coordinate plane. Students use the Distance Formula and other methods to partition directed line segments into given ratios. They then use geometric concepts of similarity and proportionality to partition directed line segments into given ratios. | • The midpoint of a line segment is the point on the segment that is equidistant from the endpoints of the line segment.<br>• The Midpoint Formula states that the midpoint between any two points on a coordinate plane, $(x_1, y_1)$ and $(x_2, y_2)$, is $((x_1 + x_2)/2, (y_1 + y_2)/2)$.<br>• The Triangle Proportionality Theorem states: "If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally." |
| **Learning Individually with MATHia or Skills Practice** | | G.SRT.2<br>G.SRT.4<br>G.SRT.5 | 4 | Students watch an animation that demonstrates that when figures are similar, a series of rigid motions and dilations can map one figure onto the other. They identify similar figures and determine corresponding side lengths and corresponding angle measures. Students then calculate corresponding parts of similar triangles, both in and out of context. They use the AA Similarity Theorem, SSS Similarity Theorem, and SAS Similarity Theorem to prove the Parallel Segment Proportionality Theorem and Triangle Midsegment Theorem. | |

*Pacing listed in 45-minute days
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CARNEGIE LEARNING

## Topic 2: Trigonometry

Students begin by exploring the relationships between the side lengths of similar right triangles. They use this investigation to define the trigonometric ratios: sine, cosine, and tangent ratios and the corresponding cosecant, secant, and cotangent ratios. They use these ratios and their inverses to solve real-world problems for unknown side lengths and angle measures. Finally, students explore the complementary relationships involved with trigonometric ratios and use them to solve problems.

**Standards:** G.SRT.6, G.SRT.7, G.SRT.8 **Pacing:** 10 Days



*Pacing listed in 45-minute days

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| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **The Sine Ratio**<br>Sine Ratio, Cosecant Ratio, and Inverse Sine | G.SRT.6<br>G.SRT.8 | 1 | The terms *sine*, *cosecant*, and *inverse sine* are introduced. A real-world context is given for determining the sine ratio in right triangles. Students conclude that as the acute angle increases in measure, the sine ratio increases in value while the cosecant value decreases, and the value of sine will always be less than 1 because the hypotenuse—the denominator in the sine ratio—is the longest side of the right triangle. When the inverse sine is introduced, students use their calculators to solve for the measure of an acute angle in a right triangle. | • In a right triangle, the ratio (side opposite to reference angle)/ hypotenuse increases as the reference angle increases.<br>• The sine (sin) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the hypotenuse.<br>• The cosecant (csc) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is opposite the angle.<br>• The inverse sine (or arcsine) of $x$ is the measure of an acute angle whose sine is $x$. |
| 4 | **The Cosine Ratio**<br>Cosine Ratio, Secant Ratio, and Inverse Cosine | G.SRT.6<br>G.SRT.8 | 1 | The terms *cosine*, *secant*, and *inverse cosine* are introduced. A real-world context is given for determining the cosine ratio in right triangles. Students conclude that as the acute angle increases in measure, the cosine ratio decreases in value while the secant value increases. Additionally, the value of cosine will always be less than 1 because the hypotenuse—the denominator in the sine ratio—is the longest side of the right triangle. When the inverse cosine is introduced, students use their calculators to solve for the measure of an acute angle in a right triangle. | • The cosine (cos) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the hypotenuse.<br>• The secant (sec) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is adjacent to the angle.<br>• The inverse cosine (or arccosine) of $x$ is the measure of an acute angle whose cosine is $x$. |
| 5 | **We Complement Each Other**<br>Complement Angle Relationships | G.SRT.7<br>G.SRT.8 | 1 | Students explore the complementary relationships involved with trigonometric ratios and use them to solve application problems. The Pythagorean Theorem in conjunction with complementary relationships is used to determine the value of the six trigonometric ratios of a 45° angle, a 30° angle, and a 60° angle. | • When $\angle A$ and $\angle B$ are acute angles in a right triangle, $\sin \angle A = \cos \angle B$ and $\cos \angle A = \sin \angle B$.<br>• When $\angle A$ and $\angle B$ are acute angles in a right triangle, $\csc \angle A = \sec \angle B$ and $\sec \angle A = \csc \angle B$.<br>• When $\angle A$ and $\angle B$ are acute angles in a right triangle, $\tan \angle A = \cot \angle B$ and $\cot \angle A = \tan \angle B$. |
| **Learning Individually with MATHia or Skills Practice** | | | 4 | Students use similar triangles to define and understand the trigonometric ratios sine, cosine, and tangent. They then explore the sine, cosine, and tangent and estimate these ratios using an interactive Explore Tool with a unit circle, as well as describing the ratios as percentages of different lengths. Students solve problems in various contexts using the trigonometric ratios and the Explore Tool. They then explore the relationship between the sine and cosine of complementary angles. Students calculate the measures of sides and angles of right triangles using trigonometric ratios, the Pythagorean Theorem, and/or the Triangle Sum Theorem in both real-world and mathematical problems. | |

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CARNEGIE LEARNING

## Topic 3: Circles and Volume

Students use similarity transformations to prove that all circles are similar. The term arc length is defined as a proportion of the circumference. Students establish that the length of the arc intercepted by an angle is proportional to the radius and define the radian measure of the angle as the constant of proportionality. They define the area of a sector in terms of a proportion of the area of the circle that contains it. They use these formulas to solve real world problems. Then students analyze the three-dimensional solid formed when translating a figure along a line segment and use Cavalieri's Principles to derive the volume formulas for prisms and cylinders. Limit arguments and Cavalieri's Principles are used to derive the volume formulas for pyramids and cones. Students use the formulas for the volume and surface area of geometric solids to solve real-world problems.

**Standards:** G.C.1, G.C.2, G.C.3, G.C.5, G.GMD.1, G.GMD.3, G.SRT.5, A.CED.4 **Pacing:** 11 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **All Circles Great and Small**<br><br>Similarity Relationships in Circles | G.C.1<br>G.C.5 | 2 | Students informally explore a different way of measuring lengths on a circle using radius measures. They then use similarity transformations to demonstrate that all circles are similar. Students use the fact that all circles are similar to demonstrate that $C = 2\pi r$. *Arc length* is defined as a portion of the circumference of a circle and it is distinguished from arc measure. Students convert between arc measure and arc length and solve problems involving arc lengths. Finally, they explore radian measures, formalizing the ideas they explored in the opening of the lesson. | • The radius of a circle, $r$, maps onto the circumference of the circle $2\pi$ times.<br>• All circles are similar figures.<br>• There is a proportional relationship between the measure of an arc length of a circle, $s$, and the circumference of the circle.<br>• The formula for arc length can be written as $s = m/(360°)(2\pi r)$ where $s$ is the arc length and $m$ is the central angle measure.<br>• One radian is the measure of a central angle whose arc length is the same as the radius of the circle.<br>• The formula for arc length can be written $s = \theta r$, where $s$ is the arc length and $\theta$ is the central angle measure in radians.<br>• When converting degrees to radians, multiply a degree measure by $\pi/180°$ and when converting radians to degrees, multiply a radian measure by $180°/\pi$. |
| 2 | **A Slice of Pi**<br><br>Sectors and Segments of a Circle | G.C.5 | 1 | The terms *sector of a circle* and *segment of a circle* are defined. Students explore and describe methods for determining the area of a sector and the area of a segment of a circle. The formula for each is stated and students apply them to solve problem situations. The formulas for linear and angular velocity are presented and are used to solve problems. | • A sector of a circle is a region of the circle bounded by two radii and the included arc.<br>• The area for the sector of a circle can be determined by multiplying the area of the circle, $A = \pi r^2$, by the fraction $m/360°$, where $m$ represents the central angle measure of the sector.<br>• The area for a sector of a circle can be determined by the formula $A_{sector} = m/360° (\pi r^2)$.<br>• The segment of a circle is a region of the circle bounded by a chord and the included arc. Each segment of a circle can be associated with a sector of the circle.<br>• The strategy for calculating the area of a segment of a circle is to calculate the area of the sector associated with the segment and from that, subtract the area of the triangle within the sector formed by the two radii and the chord or $A_{segment} = A_{sector} – A_{triangle}$.<br>• The area for a segment of a circle can be determined by the formula $A_{segment} = m/360° (\pi r^2) – 1/2(b)(h)$. |

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| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **Cakes and Pancakes**<br>Building Three-Dimensional Figures | G.GMD.1 | 1 | Students stack congruent two-dimensional figures to create three-dimensional figures. Models of two-dimensional figures are translated in space using isometric dot paper and other drawings. Students analyze the three dimensional solid figures associated with the set of all the translations along a line segment. Stacking congruent figures creates prisms and cylinders and develops an informal argument for the corresponding volume formulas. Finally, students use Cavalieri's Principles to show that areas and volumes remain the same when bases and heights remain the same. | • Rigid motion is used in the process of redrawing two-dimensional plane figures as three-dimensional solids.<br>• Models of three-dimensional solids are formed using translations of plane figures through space.<br>• Models of two-dimensional plane figures are stacked to create models of three-dimensional solids.<br>• The volume formula for a cylinder is $V = \pi r^2 h$, where $V$ is the volume, $r$ is the length of the radius of the base, and $h$ is the height of the cylinder.<br>• Cavalieri's Principle for area states that if the lengths of one-dimensional slices—just line segments—of two figures are the same, then the figures have the same area.<br>• Cavalieri's Principle for volume states that, given two solids included between parallel planes, if every plane cross section parallel to the given planes has the same area in both solids, then the volumes of the solids are equal. |
| 4 | **Get to the Point**<br>Building Volume and Surface Area Formulas for Pyramids, Cones, and Spheres | G.GMD.1<br>G.GMD.3<br>G.MG.1<br>G.MG.2 | 3 | Students consider what happens when you stack similar figures instead of congruent figures. In a hands-on activity, students relate the volume of a pyramid to the volume of a cube with the same base. They use a spreadsheet to explore the limit of the ratio as the prisms and pyramids grow and observe that the volume of a pyramid is one-third the volume of a cube with the same base. They then use Cavalieri's Principle to derive the formula for the volume of a cone. Students solve a problem about the volume of two cylinders with the same lateral area and then determine the lateral and total surface areas of various three-dimensional solid figures. Students are introduced to the formulas for the volume and surface area of a sphere. Finally, students solve a variety of real-world problems using reasoning and the volume and surface area formulas. | • The volume formula for a pyramid is $V = (1/3)Bh$, where $V$ is the volume, $B$ is the area of the base, and $h$ is the height of the pyramid.<br>• The volume of a pyramid is one third the volume of a prism with the same base area and height<br>• The volume formula for a cone is $V = (1/3)\pi r^2 h$, where $V$ is the volume, $r$ is the radius of the base, and $h$ is the height of the cone.<br>• The volume of a cone is one third the volume of a cylinder with the same base area and height.<br>• The lateral surface area of a three-dimensional figure is the sum of the areas of its lateral faces. The total surface area of a three-dimensional figure is the sum of its bases and lateral faces.<br>• The volume formula for a sphere is $V = (4/3)\pi r^3$, where $V$ is the volume, and $r$ is the radius of the sphere.<br>• The total surface area of a sphere is $4\pi r^2$. |
| **Learning Individually with MATHia or Skills Practice** | | G.C.2<br>G.C.5<br>G.GMD.3 | 4 | Students explore the difference between the degree measure of an arc and the length of an arc. They then practice calculating the fraction of a circle's circumference that an arc occupies and writing an expression that can be used to calculate an arc length. Students then calculate the arc length given the radius or diameter of the circle. Next, they relate the arc length to the circle's radius and are introduced to radians and the theta symbol. Finally, students practice determining different measurements of a circle using the formula $\theta = s/r$. Students calculate the measure of an arc or an angle using the Interior Angles of a Circle Theorem and Exterior Angles of a Circle Theorem. Students are given the definition of a sector of a circle and practice identifying sectors. They then work through an example that develops the formula for determining the area of a sector of a circle before using the formula to find areas of different sectors of circles. Students rotate two-dimensional figures about an axis to create three-dimensional shapes and relation the dimension of the plane figure to the solid. They differentiate between right and oblique solids and then create solids by stacking content of similar shapes. They then use mathematical and real-world objects to determine the volume of cylinders, pyramids, cones, and spheres. | |

*Pacing listed in 45-minute days
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CARNEGIE LEARNING

# 3 Exploring Functions

## Pacing: 37 Days

### Topic 1: Functions Derived From Linear Relationships

Students connect the absolute value of a number to the absolute value of a function. They examine the graphs of absolute value functions, noting the reflection of points with negative y-values across the *x*-axis. The visual representation allows students to see that each output corresponds to two input values. From this intuition, students solve absolute value equations and inequalities. Students extend their understanding of transforming linear functions to transform absolute value functions, now considering the effect of adding a constant to the argument. Linear piecewise functions are introduced, and students analyze the restricted domain of each piece. They then write absolute value functions as piecewise functions to connect the two concepts. Step functions are introduced as a special case of piecewise functions. Finally, students reflect functions across $y = x$ to generate the inverse of a linear function.

**Standards:** F.IF.4, F.IF.7b, F.BF.3, A.CED.2, A.CED.3, A.REI.11 **Pacing:** 12 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **Putting the V in Absolute Value**<br><br>Defining Absolute Value Functions and Transformations | F.IF.7b<br>F.BF.3 | 2 | Students model absolute value functions and their transformations on a human coordinate plane. They explore and analyze different transformations of absolute value functions, their graphs, and equations, and summarize the effects of these transformations. By transforming absolute value functions, students distinguish between the effects of changing values inside the argument of the function vs. changing values outside the function. | • An absolute value function is a function of the form $f(x) = \|x\|$.<br>• A function $g(x)$ of the form $g(x) = f(x) + D$ is a vertical translation of the function $f(x)$. The value $\|D\|$ describes the number of units the graph of $f(x)$ is translated up or down. If $D > 0$, the graph is translated up; if $D < 0$, the graph is translated down.<br>• A function $g(x)$ of the form $g(x) = Af(x)$ is a vertical dilation of the function $f(x)$. For $\|A\| > 1$, the graph is vertically stretched by a factor of $A$ units; for $0 < \|A\| < 1$, the graph vertically compresses by a factor of $A$ units. For $A < 0$, the graph also reflects across the *x*-axis.<br>• A function $g(x)$ of the form $g(x) = f(x - C)$ is a horizontal translation of the function $f(x)$. The value $\|C\|$ describes the number of units the graph of $f(x)$ is translated right or left. If $C > 0$, the graph is translated to the right; if $C < 0$, the graph is translated to the left.<br>• Transforming a function by changing the *A*- or *D*-values affects the output of the function, *y*. Transforming a function by changing the *C*-value affects the input of the function, *x*. |
| 2 | **Play Ball!**<br><br>Absolute Value Equations and Inequalities | A.CED.2<br>A.CED.3<br>A.REI.11<br>F.IF.7b | 2 | Students create a linear absolute value function to model a scenario and use the graph of the function to estimate solutions in the context of the problem situation. They then solve a linear absolute value equation algebraically by first rewriting it as two separate linear equations. One equation represents the case where the value of the expression inside the absolute value is positive, and the second represents the case where it is negative. Students model a scenario with a linear absolute value inequality and its corresponding graph. Finally, they solve linear absolute value inequalities algebraically by first rewriting them as equivalent compound inequalities. | • Linear absolute value equations have 0, 1, or 2 solutions. The solution set for linear absolute value inequalities may contain all real numbers, a subset of the real numbers represented by a compound inequality, or no solutions.<br>• Linear absolute value inequalities can be rewritten as equivalent compound inequalities.<br>• Linear absolute value equations and inequalities can be used to represent real-world situations. |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

CARNEGIE LEARNING

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **I Graph in Pieces**<br><br>Linear Piecewise Functions | F.IF.4<br>F.IF.7b | 2 | Students develop a piecewise function from a scenario. The terms *piecewise function* and *linear piecewise* function are defined. They analyze a piecewise graph and write a scenario and piecewise function to represent the graph. Students analyze statements that correspond to different pieces of the graphed function. They then write a scenario that can be modeled with a piecewise function, graph a partner's scenario, and work with their partner to write a piecewise function for each. | • A piecewise function is a function that can be represented by more than one function, where each function corresponds to a specific part of the domain.<br>• To write a piecewise function, you must write the equation and domain for each piece of the function.<br>• Piecewise functions can be represented by scenarios, equations, tables, and graphs.<br>• A linear absolute value function can be constructed using a piecewise function.<br>• Graphing technology can be used to model piecewise functions. |
| 4 | **Step by Step**<br><br>Step Functions | F.IF.7b | 2 | Students analyze the graph of a special piecewise function. The terms *discontinuous graph* and *step function* are defined, and students interpret those definitions through examination of graphs. Students are then given a context and must provide both the piecewise function and the graph that models it, explaining why the function is a step function. Next, they are introduced to the greatest integer function (floor function) and least integer function (ceiling function) through their definitions, notation, meaning of individual values, graphs and real-world examples. | • A discontinuous graph is a graph that is continuous for some values of the domain with at least one disjoint area between consecutive $x$-values.<br>• A step function is a piecewise function on a given interval whose pieces are discontinuous constant functions.<br>• The greatest integer function, also known as the floor function, is a special type of step function. The greatest integer function $G(x) = \lfloor x \rfloor$ is defined as the greatest integer less than or equal to $x$.<br>• The least integer function, also known as the ceiling function, is a special type of step function. The least integer function $L(x) = \lfloor x \rfloor$ is defined as the least integer greater than or equal to $x$. |
| 5 | **A Riddle Wrapped in a Mystery**<br><br>Inverses of Linear Functions | F.BF.4a | 1 | Students use a table of values to determine the inverse of a given problem situation. The term inverse of a function is defined and students are shown how to algebraically determine the inverse of a function. They then create the graph of the inverse of a linear function by reflecting the original function across the line $y = x$ using patty paper. The term one-to-one function is defined, and students determine whether given functions are one-to-one functions. | • Inverse functions can be determined algebraically and graphically.<br>• The inverse of a function is determined by replacing $f(x)$ with $y$, switching the $x$ and $y$ variables, and solving for $y$.<br>• The graph of the inverse of a function is a reflection of that function across the line $y = x$.<br>• A one-to-one function is a function in which its inverse is also a function.<br>• For a one-to-one function $f(x)$, the notation for its inverse is $f^{-1}(x)$. |
| **Learning Individually with MATHia or Skills Practice** | | A.CED.3<br>F.IF.7b | 3 | Students solve and graph simple absolute value equations in one variable. They use graphical representations to solve absolute value inequalities and learn to write equivalent compound inequalities for absolute value inequalities. Students analyze a linear piecewise function from a real-world scenario. They then match sketches of graphs of linear piecewise functions to given scenarios and identify the graph of a linear piecewise function given an equation. Students identify the domain in both non-continuous and continuous piecewise functions and determine the equation given a problem situation and a graph. Students identify the step function that represents a given problem situation and graph. | |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

## Topic 2: Exponentials

Students expand on their understanding of exponential functions. They examine the structure of exponential expressions, concluding that $2^{(1/2)} = \sqrt{2}$ and $2^{(1/3)} = \sqrt[3]{2}$. Students expand their experiences with function transformations, adding horizontal dilations and horizontal reflections to their repertoire. They rewrite exponential functions with *B*-value transformations using the rules of exponents. Finally, students combine linear and exponential function types, noting the characteristics of the resulting exponential function.

**Standards:** N.RN.1, N.RN.2, N.RN.3, F.IF.8b, A.SSE.1b, A.SSE.3c **Pacing:** 9 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **Got Chills ... They're Multiplyin'**<br>Exponential Functions and Rational Exponents | N.RN.1<br>N.RN.2<br>N.RN.3<br>F.BF.1a | 2 | Students explore the output values for exponential functions for input values between two integers, and they convert between expressions with rational exponents and those written in radical notation. Students construct a large graph of the function $f(x) = 2^x$ given a situation; they then use this graph, the Power Rules, and the idea of a constant ratio to determine output values for the exponential function when the input values are non-integers. Students use this exploration to connect expressions with rational exponents to those in radical notation. Finally, they practice converting between expressions with rational exponents and those in radical notation and make generalizations about the constant ratio for exponential functions. | • If the difference in the input values is the same, an exponential function shows a constant ratio between output values, no matter how large or how small the gap between input values.<br>• Multiple representations such as tables, equations, and graphs are used to represent exponential problem situations.<br>• Properties of powers can be used to rewrite numeric and algebraic expressions involving integer and rational exponents.<br>• Because rational exponents can be rewritten as radicals, the properties of powers apply to radical expressions, as well.<br>• A rational expression of the form $a^{(m/n)}$ can be written as a radical expression $\sqrt[n]{a^m}$ or $(\sqrt[n]{a})^m$. |
| 2 | **Turn That Frown Upside Down**<br>Growth and Decay Functions | A.SSE.1b<br>F.IF.8b<br>F.LE.1c<br>F.LE.3<br>F.LE.5 | 2 | This lesson uses scenarios that can be modeled by increasing or decreasing exponential functions. Students compare linear and exponential functions in the context of simple interest and compound interest situations. Next, they identify the values in the exponential equation that indicate whether the function is a growth or decay function, and they apply this reasoning in context. Finally, for a situation modeled by an exponential decay function, students write the function and sketch its graph, then use the graph to answer a question about the problem situation. | • Simple interest can be represented by a linear function. Compound interest can be represented by an exponential function.<br>• When an exponential function of the form $f(x) = ab^x$ with $a > 0$ has a *b*-value greater than 1, it represents an increasing or growth function. The *b*-value may be represented by $(1 + r)$, where $r$ is the rate of growth.<br>• When an exponential function of the form $f(x) = ab^x$ with $a > 0$ has a *b*-value between 0 and 1, it represents a decreasing or decay function. The *b*-value may be represented by $(1 - r)$, where $r$ is the rate of decay. |
| 3 | **Just So ... Basic**<br>Horizontal Dilations of Exponential Functions | A.SSE.3c<br>F.BF.3 | 1 | Students review transformational forms for exponential functions, along with the exponential function from Lesson 1 that models the frequencies of notes on a keyboard. Students connect the work they did in the previous lesson with a change in the *B*-value of the transformational form of the function. Given a function that represents an annual increase in a mutual fund, students use the properties of exponents to rewrite the function to reveal approximate equivalent rates for the monthly and quarterly increases. Students graph different transformations of the basic exponential function and describe how increasing or decreasing the *B*-value affects various situations modeled by exponential functions. | • Transforming exponential functions into equivalent forms can reveal different properties of the quantities represented.<br>• Changing the *B*-value of an exponential function in transformational form creates a horizontal dilation of the function.<br>• Changing the *B*-value of an exponential function can be described in terms of the context which is modeled by the function. |

*Pacing listed in 45-minute days

† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 4 | **Saving Strategies**<br>Modeling with and Combining Function Types | F.BF.1b | 1 | Students add constant functions to exponential functions—as equations, graphically, and in context—and explain the meaning of the constant function and its effect on the exponential function in terms of the context. Students use a savings context to investigate how exponential functions can be combined with constant functions to produce a sum function that is a translation of the original exponential function. Students model the cooling of water at a given room temperature using what they know about transformations of exponential functions and adding constant functions to exponential functions. Students end the lesson by sketching the sum of exponential and constant functions. | • An exponential function and a constant function can be added to create a third function that is the sum of the two functions, resulting in a graph that is a vertical translation of the original exponential function.<br>• When a constant function is added to an exponential function, the *D*-value and asymptote of the exponential function are affected.<br>• The sum of an exponential and constant function can be used in context to model a real-world situation. |
| **Learning Individually with MATHia or Skills Practice** | | N.RN.1<br>N.RN.2<br>F.BF.3<br>F.LE.1b<br>F.LE.1c | 3 | Students compare linear and exponential functions and their graphs in the context of simple and compound interest. They solve problems related to the independent and dependent variables of both linear and exponential functions using the graphs and equations. Students then watch two different animations: one shows a model of exponential growth and one shows a model of exponential decay. They analyze how to recognize the difference between the two exponential models, then interpret exponential functions using scenarios of population increase or decrease. Students learn the names of the components of radical notation (radical, radicand, index and $n^{th}$ root). They use the properties of powers to make sense of the fact that $x^{(1/2)} = \sqrt{x}$. Students expand their understanding of rational exponents to include making sense of fractional exponents with a numerator other than 1 and negative exponents. Given various expressions with rational exponents, they select an equivalent radical expression and vice versa. Students use four animations, demonstrating the different ways of transforming an exponential function, to investigate how changing the equation for an exponential function changes the graph of the function. They answer questions related to horizontal and vertical translations and dilations of exponential functions. They transform graphs of linear and exponential functions in a variety of ways: using vertical and horizontal reflections, vertical and horizontal dilations, and vertical and horizontal translations. Given a representation of a transformed function, students determine how the basic linear and exponential functions were transformed to create the new functions. | |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

CARNEGIE LEARNING

## Topic 3: Introduction to Quadratic Functions

Students explore the structure of quadratic functions through four real-world situations. They learn key characteristics of quadratic functions through inspection of graphs and equations: intercepts, absolute maximum/minimum, vertex, axis of symmetry, domain, range, and intervals of increase and decrease. Students transform functions using the function transformation form $y = A \cdot f(B(x - C)) + D$ and learn how coordinates are affected; any point $(x, y)$ on the graph $y = f(x)$ maps to the point $((1/B)x + C, Ay + D)$ on the graph $y = A \cdot f(B(x - C)) + D$. Given quadratic functions represented in different forms (table, graph, equation, or scenario), students compare key characteristics.

**Standards:** A.REI.10, A.REI.11, F.IF.4, F.IF.5, F.IF.6, F.IF.7a, F.IF.8a, F.IF.9, A.SSE.1a, A.SSE.3a, A.APR.3, F.BF.3, A.CED.4, F.LE.3 **Pacing:** 16 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **Up and Down or Down and Up**<br>Exploring Quadratic Functions | A.REI.10<br>A.REI.11<br>F.IF.4<br>F.IF.5<br>F.IF.7a | 2 | Students are introduced to quadratic functions and their growth pattern through a sequence of pennies. They are then provided four different contexts that can be modeled by quadratic functions. For each function, students address the key characteristics of the graphs and interpret them in terms of the context. They also compare the domain and range of the functions and the context they represent. The first context involves area and is used to compare and contrast linear and quadratic relationships. The second context involves handshakes and has the student write the function. The third context involves catapulting a pumpkin. Students analyze this function written in general form. The final context involves revenue and demonstrates how a quadratic function can be written as the product of two linear functions. | • Quadratic functions can be used to model certain real-world situations.<br>• The graph of a quadratic function is called a parabola.<br>• A parabola is a smooth curve with reflectional symmetry.<br>• A parabola has an absolute maximum or absolute minimum point and an interval where it is increasing and an interval where it is decreasing.<br>• A parabola has one $y$-intercept and at most two $x$-intercepts.<br>• The domain of a quadratic function is the set of all real numbers. The range is a subset of the real numbers that is limited based upon the $y$-coordinate of the absolute maximum or absolute minimum point. |
| 2 | **Endless Forms Most Beautiful**<br>Key Characteristics of Quadratic Functions | A.SSE.1a<br>A.SSE.3a<br>A.APR.3<br>F.IF.4<br>F.IF.6<br>F.IF.7a<br>F.IF.8a | 3 | Students revisit the four scenarios from the previous lesson as a way to introduce equivalent quadratic equations with different structures to reveal different characteristics of their graphs. They learn that a table of values represents a quadratic function if its second differences are constant. Students analyze the effect of the leading coefficient on whether the parabola opens up or down. They identify the axis of symmetry and vertex for graphs using the equations in each form. Finally, students determine the $x$- and $y$-intercepts along with intervals of increase and decrease, using a combination of technology, symmetry, and equations. | • A table of values representing a quadratic function has constant second differences.<br>• A quadratic function may be written in general form, $f(x) = ax^2 + bx + c$, where $a \neq 0$, and in factored form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$.<br>• When the $a$-value of a quadratic function in general form or factored form is positive, the graph opens upward and has an absolute minimum, and when the leading coefficient is negative, the graph opens downward and has an absolute maximum.<br>• The vertex (or maximum/minimum) of a parabola lies on its axis of symmetry. The axis of symmetry can be determined by the formula $x = (r_1 + r_2)/2$ from the factored form or by $x = -b/2a$ from the general form of a quadratic equation.<br>• In factored form, $f(x) = a(x - r_1)(x - r_2)$, the values of $(r_1, 0)$ and $(r_2, 0)$ are the $x$-intercepts of the quadratic function. In general form, $f(x) = ax^2 + bx + c$, $(0, c)$ is the $y$-intercept of the quadratic function. |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

CARNEGIE LEARNING

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **More Than Meets the Eye**<br><br>Transformations of Quadratic Functions | A.SSE.3a<br>F.BF.3 | 2 | Students explore a variety of different transformations of quadratic functions, including vertical translations, horizontal translations, vertical dilations and reflections, and horizontal dilations and reflections. For each transformation, students sketch graphs of the transformation, compare characteristics of the transformed graphs with those of the graph of the basic function, and write the transformations using coordinate notation. Students write quadratic equations in vertex form using the coordinates of the vertex and another point on the graph and in factored form using the zeros and another point on the graph. | • The general transformation equation is $y = Af(B(x - C)) + D$, where the $A$-value describes a vertical dilation or reflection, the $B$-value describes a horizontal dilation or reflection, the $C$-value describes a horizontal translation, and the $D$-value describes a vertical translation.<br>• Given $f(x) = x^2$ as the basic quadratic function, reference points can be used to graph $y = Af(B(x - C)) + D$ such that any point $(x, y)$ on $f(x)$ maps to the point $((1/B)x + C, Ay + D)$.<br>• The vertex form for a quadratic function is $f(x) = a(x - h)^2 + k$, where $(h, k)$ is the location of the vertex, and the sign of a indicates whether the parabola opens upward or downward.<br>• A function written in equivalent forms can reveal different characteristics of the function it defines.<br>• You can write a quadratic function in vertex form if you know the coordinates of the vertex and another point on the graph.<br>• You can write a quadratic function in factored form if you know the zeros and another point on the graph. |
| 4 | **You Lose Some, You Lose Some**<br><br>Comparing Functions Using Key Characteristics and Average Rate of Change | A.SSE.1a<br>A.CED.4<br>F.IF.9<br>F.LE.3 | 1 | Students compare quadratic functions in standard form, factored form, and vertex form, then analyze the properties of each form. Students then answer questions to compare linear, quadratic, and exponential functions. They compute average rates of change for the functions across different intervals and then compare the change in the average rates of change across the different intervals. Quadratic equations in different forms are compared by identifying key characteristics of their representations. | • The average rate of change of any function over an interval is the slope of a linear function passing through the beginning and end points of the interval.<br>• A quantity increasing exponentially eventually exceeds a quantity increasing linearly or quadratically.<br>• The average rate of change of an increasing exponential function will eventually exceed the average rate of change for an increasing linear and quadratic function.<br>• A function written in equivalent forms can reveal different characteristics of the function it defines.<br>• Quadratic equations in different forms are compared by identifying key characteristics of their representations or by changing one representation to match the other for comparison purposes. |
| **Learning Individually with MATHia or Skills Practice** | | A.CED.1<br>F.IF.3<br>F.IF.4 | 8 | Students complete a table of values and graph from a scenario represented by a quadratic model. Students construct the quadratic function for the scenario as a product of a monomial and a binomial or a product of two binomials. Students use an interactive Explore Tool to investigate how a vertical motion graph changes when the different values in the vertex, factored, and general form of the quadratic function change. They then use vertical motion graphs to identify the maximum, $x$-intercepts, $y$-intercept, domain, range, axis of symmetry, and vertex of a quadratic function. Students transform linear and quadratic functions. Given a representation of a transformed function, students determine how the basic linear and quadratic functions were transformed to create the new functions. | |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

CARNEGIE LEARNING

# 4 Seeing Structure

## Pacing: 41 Days

## Topic 1: Solving Quadratic Equations

Students begin by solving simple quadratic equations in the form $x^2 = n$ using horizontal lines and the Properties of Equality, skills they learned in middle school. They recognize that the solutions to a quadratic equation are equidistant to the axis of symmetry on a graph of the function. They use these tools to solve increasingly complex quadratic equations, each time making a connection to the location of the solutions on a graphical representation. When they encounter an equation in general form, students learn to use the Zero Product Property to solve via factoring. For trinomials that cannot be factored, students learn to complete the square and to use the Quadratic Formula. In deriving the Quadratic Formula, students recognize that its structure supports the fact that all solutions are equidistant from the axis of symmetry for any quadratic equation. Students are provided multiple opportunities to solve quadratic equations using efficient methods.

**Standards:** N.RN.2, N.RN.3, N.CN.1, A.SSE.1a, A.SSE.1b, A.SSE.2, A.SSE.3a, A.SSE.3b, A.APR.1, A.REI.4, A.REI.4a, A.REI.4b, A.REI.10, F.IF.8a **Pacing:** 21 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **This Time, with Polynomials**<br><br>Adding, Subtracting and Multiplying Polynomials | A.SSE.1a<br>A.APR.1 | 2 | Students are introduced to polynomials and identify the terms and coefficients of polynomials. Students sort polynomials by the number of terms, rewrite in general form if possible, and identify the degree. Students add and subtract polynomial functions algebraically and graphically and then determine that polynomials are closed under addition and subtraction. Students use area models and the Distributive Property to determine the product of binomials. They explore special products and are introduced to the terms difference of two squares and perfect square trinomial. | • A polynomial is a mathematical expression involving the sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable is the sum of terms of the form $ax^k$, where $a$ is any real number and $k$ is a non-negative integer. In general, a polynomial is of the form $a_1x^k + a_2x^{(k-1)} + \ldots + a_nx^0$.<br>• Polynomials with only one term are monomials. Polynomials with exactly two terms are binomials. Polynomials with exactly three terms are trinomials.<br>• Polynomials can be added, subtracted, and multiplied using algebraic operations.<br>• Polynomials are closed under addition, subtraction, and multiplication.<br>• The difference of two squares and perfect square trinomials are special products that can be recognized when multiplying binomials. |
| 2 | **Solutions, More or Less**<br><br>Representing Solutions to Quadratic Equations | N.RN.2<br>A.SSE.2<br>A.SSE.3a<br>A.REI.4b<br>A.REI.10 | 2 | Students use the Properties of Equality and square roots to solve simple quadratic equations. They express solutions in terms of the distance from the axis of symmetry to the parabola. Students identify double roots, estimate square roots, and extract perfect roots from the square roots of products. They show graphically that a quadratic function is the product of two linear functions with the same zeros. Students then use the Zero Product Property to explain that the zeros of a quadratic function are the same as the zeros of its linear factors. Finally, they rewrite any quadratic in the form $f(x) = ax^2 - c$ as the product of two linear factors. | • Every whole number has two square roots, a positive principal square root and a negative square root.<br>• A quadratic function is a polynomial of degree 2. Thus, a quadratic function has two zeros or two solutions at $f(x) = 0$. If both solutions are the same, the quadratic function is said to have a double zero.<br>• The $x$-coordinates of the $x$-intercepts of a graph of a quadratic function are called the zeros of the quadratic function. The zeros are called the roots of the quadratic equation.<br>• The real solutions to a quadratic equation can be represented as the $x$-value of the axis of symmetry plus or minus a constant.<br>• The Zero Product Property states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero. Therefore, the zeros of a quadratic function are the same as the zeros of its linear factors.<br>• Any quadratic function in the form $f(x) = ax^2 - c$ can be rewritten as the product of two linear factors, $(\sqrt{a}x + \sqrt{c})(\sqrt{a}x - \sqrt{c})$. |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **Transforming Solutions**<br><br>Solutions to Quadratic Equations in Vertex Form | A.SSE.2<br>A.SSE.3a | 2 | Students solve quadratic equations that are squares of binomials, recognizing them as horizontal translations of the function. They use the Properties of Equality and square roots to solve quadratic equations of the form $y = a(x - c)^2$ and determine how the dilation affects the solutions. Finally, students solve quadratic functions of the form $y = a(x - c)^2 + d$ and determine how the translation affects the solutions. Students learn that a quadratic function can have one unique real zero, two real zeros, or no real zeros, and how the number of real zeros relates to the graph of the function. | • The solutions to a quadratic equation can be represented as the $x$-value of the axis of symmetry plus or minus a constant.<br>• A quadratic function written in the form $f(x) = a(x - h)^2 + k$, where $a \neq 0$, is in vertex form.<br>• The solutions for a quadratic equation of the form $y = (x - c)^2$ are $x = c \pm \sqrt{y}$.<br>• The solutions for a quadratic equation of the form $y = a(x - c)^2$ are $x = c \pm \sqrt{(y/a)}$.<br>• The solutions for a quadratic equation of the form $y = a(x - c)^2 + d$ are $x = c \pm \sqrt{((y - d)/a)}$. |
| 4 | **The Missing Link**<br><br>Factoring and Completing the Square | A.SSE.3b<br>A.REI.4a<br>F.IF.8a | 2 | Students recall how to factor out the GCF from different polynomials. They follow examples to factor quadratic trinomials, first using area models and then recognizing patterns in the coefficients. Students use the Zero Product Property to solve quadratic equations by factoring. They are then introduced to completing the square, a method they can use to convert a quadratic equation given in general form to vertex form. Students complete the square to solve quadratic equations that cannot be solved using other methods. | • One method of solving quadratic equations in the form $y = ax^2 + bx + c$ is to set the equation equal to zero, factor the trinomial expression, and use the Zero Product Property to determine the roots.<br>• Completing the square is a method for rewriting a quadratic equation in the form $y = ax^2 + bx + c$ as a quadratic equation in vertex form.<br>• When a quadratic equation in the form $y = ax^2 + bx + c$ is not factorable, completing the square is an alternative method of determining the roots of the equation.<br>• Completing the square is a useful method for converting a quadratic function written as $f(x) = ax^2 + bx + c$ to vertex form for graphing purposes and determining the maximum or minimum in problem situations.<br>• Given a quadratic equation in the form $y = ax^2 + bx + c$, the vertex of the function is located at $(x, y)$ such that $x = -b/2a$ and $y = c - (b^2)/4a$. |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 5 | **Ladies and Gents, Please Welcome the Quadratic Formula!**<br><br>The Quadratic Formula | N.RN.2<br>N.RN.3<br>N.CN.1<br>A.SSE.1b<br>A.REI.4<br>A.REI.4a<br>A.REI.4b | 3 | The first activity focuses on the graphical interpretation of x = -(b/2a) +/- (sqrt(b^2 - 4ac)/2a) as the distance from (-b/2a, 0) to each root. Students are introduced to the Quadratic Formula as a method to calculate the solutions to any quadratic equation written in general form. Students use the discriminant to determine the number and type of roots for a given function. Students learn why rational numbers are closed under addition and that the sum or product of a rational number and an irrational number is an irrational number. | • The Quadratic Formula, $x = (-b \pm \sqrt{(b^2 - 4ac)})/2a$, is a method to calculate the solutions to any quadratic equation written in general form, where $a$, $b$, and $c$ represent real numbers and $a \neq 0$.<br>• On the graph of a quadratic function, $\pm \sqrt{(b^2 - 4ac)}/2a$ is the distance from $(-b/2a, 0)$ to each root.<br>• When a quadratic equation is in the form, $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ represent real numbers and $a \neq 0$, the discriminant is $b^2 - 4ac$. When $b^2 - 4ac < 0$, the quadratic equation has no real roots. When $b^2 - 4ac = 0$, the quadratic equation has one real root. When $b^2 - 4ac > 0$, the quadratic equation has two real roots.<br>• When $b^2 - 4ac$ is a positive perfect square number, the roots are rational numbers. When $b^2 - 4ac$ is positive, but not a perfect square number, the roots are irrational numbers.<br>• Understand that equations in one number system may have solutions in a larger number system.<br>• A rational number plus a rational number equals a rational number; rational numbers are closed under addition. A rational number plus an irrational number equals an irrational number. |
| **Learning Individually with MATHia or Skills Practice** | | A.SSE.3a<br>A.SSE.3b<br>A.APR.1<br>A.APR.6<br>A.REI.4a<br>A.REI.4b<br>A.REI.11<br>F.IF.7a<br>F.IF.8a<br>F.IF.9 | 10 | Students are introduced to polynomials and use an Explore Tool to investigate combining like terms when adding polynomial expressions. They practice adding, subtracting, multiplying, and factoring quadratic expressions. Students learn to complete the square using area models and algebraically. Students differentiate among general form, factored form, and vertex form of a quadratic equation; they learn the characteristics of the graph that are apparent in each form. Students convert between the forms of a quadratic equation and sketch the corresponding graphs. Students solve quadratic equations by factoring and applying the Zero Product Property or by using the Quadratic Formula. | |

*Pacing listed in 45-minute days
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## Topic 2: Applications of Quadratics

Students analyze the parabolas of quadratic functions that do not cross the x-axis. They use Properties of Equality and square roots to solve for x and discover the need for imaginary numbers. Students are introduced to the complex number system and operate with complex numbers before solving quadratic equations with imaginary roots using the Quadratic Formula. Students then use the structure of a parabola and a given context to solve quadratic inequalities. They use what they know about solutions to functions in a graphical representation to solve systems of equations comprising a quadratic and a linear function or two quadratic functions. Given a data set, students use technology to determine a regression curve that best fits the data and to make predictions for given input values. Finally, students reflect quadratics across $y = x$ to identify the graphical inverse of a quadratic function and then learn how to determine the equation of the inverse algebraically. Because quadratic functions are not one-to-one, students restrict the domain of quadratic functions to write their inverse functions.

**Standards:** A.CED.1, A.CED.2, A.CED.3, A.REI.4, A.REI.4a, A.REI.4b, A.REI.7, A.REI.11, F.IF.7b, F.BF.4a, F.BF.4d, S.ID.6a **Pacing:** 11 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | ***i* Want to Believe**<br><br>Imaginary and Complex Numbers | N.CN.1<br>N.CN.2<br>N.CN.7<br>N.CN.8(+)<br>N.CN.9(+)<br>A.REI.4b | 2 | Students begin by analyzing a quadratic function that does not cross the *x*-axis and use the Properties of Equality and square roots to solve the corresponding equation for its roots. Students are introduced to the number *i*, imaginary roots, imaginary zeros, and the complex number system. They use a complex coordinate plane to plot complex numbers and then use the graphical representation to understand how to add, subtract, and multiply complex numbers. Students solve quadratic equations with complex solutions using any method of their choosing. Finally, the Fundamental Theorem of Algebra is introduced, and students analyze graphs to determine the number of real and imaginary roots each corresponding quadratic equation has. | • Equations with no solution in one number system may have solutions in a larger number system.<br>• The number $i$ is a number such that $i^2 = -1$.<br>• The set of complex numbers is the set of all numbers written in the form $a + bi$, where $a$ and $b$ are real numbers and $b$ is not equal to 0.<br>• The Commutative Property, the Associative Property, and Distributive Properties apply to complex numbers.<br>• Functions that do not intersect the *x*-axis have imaginary zeros.<br>• When the discriminant of a quadratic equation is a negative number, the equation has two imaginary roots.<br>• The Fundamental Theorem of Algebra states that any polynomial equation of degree n must have n complex roots or solutions. |
| 2 | **Ahead of the Curve**<br><br>Solving Quadratic Inequalities | A.CED.1<br>A.CED.2<br>A.CED.3<br>A.REI.4<br>A.REI.4a<br>A.REI.4b | 1 | Students use the graph of a vertical motion model to approximate the times when an object is at given heights. They identify regions on the graph that are less than or greater than a given height and write a quadratic inequality to represent the situation. Next, students are shown how to solve a quadratic inequality algebraically. They determine the solution set of the inequality by dividing the graph into intervals defined by the roots of the quadratic equation, and then test values in each interval to determine which intervals satisfy the inequality. Finally, with a second scenario, students write the function that represents the situation, sketch a graph of the function, and write and solve a quadratic inequality related to the solution set of the quadratic function. | • A horizontal line drawn across the graph of a quadratic function intersects the parabola at exactly two points, except at the vertex, where it intersects the parabola at exactly one point.<br>• The solution set of a quadratic inequality is determined by first solving for the roots of the quadratic equation, and then determining which interval(s) created by the roots will satisfy the inequality. A combination of algebraic and graphical methods may be the most efficient solution method.<br>• Quadratic inequalities can be used to model some real-world contexts. The effects of translations of quadratic functions can be used to make comparisons within a context. |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

CARNEGIE LEARNING

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **All Systems Are Go!**<br><br>Systems of Quadratic Equations | A.CED.2<br>A.CED.3<br>A.REI.7<br>A.REI.11 | 2 | Students are presented with a scenario that can be modeled with a quadratic and a linear equation and reason about the intersections of the two equations in the context of the problem. Next, they solve systems of equations composed of a linear equation and a quadratic equation algebraically using substitution, factoring, and the Quadratic Formula. They then verify their algebraic solutions graphically by determining the coordinates of the points of intersection. Finally, students solve a system composed of two quadratic equations using the same methods. They conclude that a system of equations consisting of a linear and a quadratic equation can have one solution, two solutions, or no solutions, while a system of two quadratic equations can have one solution, two solutions, no solutions, or infinite solutions. | • Systems of equations involving a linear equation and a quadratic equation or two quadratic equations can be solved both algebraically and graphically.<br>• A system of equations containing a linear equation and a quadratic equation may have no solution, one solution, or two solutions.<br>• A system of equations containing two quadratic equations may have no solution, one solution, two solutions, or an infinite number of solutions.<br>• The number of solutions for a system of equations depends on the number of points where the graphs of the two equations intersect.<br>• A system of equations involving a linear equation and a quadratic equation may be used to model real-world problems. |
| 4 | **Model Behavior**<br><br>Using Quadratic Functions to Model Data | F.IF.7b<br>F.BF.4a<br>F.BF.4d<br>S.ID.6a | 2 | Students determine a quadratic regression that best models a table of data. They answer questions and make predictions using the regression equation. Students then analyze another data set and determine the inverse relation of the quadratic function they wrote. Next, the same algebraic process is used to determine the inverse of a quadratic function. Students graph equations containing square roots, identify the domain and range of each graph, and determine which graphs describe functions. Using only the equation of the inverse of a function, students then determine the original function and identify its domain and range. The term *restrict the domain* is introduced. Students determine the restrictions on the domain of a quadratic function based on the problem situation and graph the function with the restricted domain. They write the equation for the inverse function and interpret it with respect to the problem situation. Finally, students determine whether certain types of functions are one-to-one functions. | • Some data in context can be modeled by a quadratic regression equation. The regression equation can be used to make predictions; however there may be limitations on the domain depending on the context.<br>• To determine the inverse of a function, replace $f(x)$ with $y$, switch the $x$ and $y$ variables, then solve for $y$.<br>• When a problem requires using a given function to determine the independent quantity when a dependent quantity is given, determining the inverse of the original function may be a more efficient way to handle the situation.<br>• The inverse of a function may or may not be a function. A function is a one-to-one function if its inverse is also a function.<br>• To restrict the domain of a function means to define a new domain for the function that is a subset of the original domain. |
| **Learning Individually with MATHia or Skills Practice** | | F.BF.1b<br>F.BF.3<br>F.BF.4<br>S.ID.6a | 4 | Students use equations of quadratic regression models, the solver, and graphs to answer questions. They then watch an animation about operating with functions on the coordinate plane before adding and subtracting constant functions, linear functions, and a linear and a quadratic function. Given two functions in function notation, students determine the sum or difference of the functions and verify the sum or difference by evaluating the new function at a given value. Given the graphs of two relations, they decide if the relations are inverses. Students determine the equation of the inverse function for a given function and use composition of functions to verify that the functions are inverses. | |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

CARNEGIE LEARNING

## Topic 3: Circles on a Coordinate Plane

Students begin by recalling the Pythagorean Theorem and using it to write equations of circles on a coordinate plane. They complete the square to convert between general and standard forms of equations when necessary to determine the radius and center of a given circle. Students then use the Pythagorean Theorem, the Distance Formula, and symmetry to verify whether a point lies on a given circle. They then incorporate the trigonometric ratios—considering right triangles formed by the radius of the unit circle—to develop the Pythagorean identity sin^2 θ + cos^2 θ = 1. This is used to calculate the value of trigonometric ratios for angles in given quadrants of the coordinate plane. Finally, students use concentric circles around a focus and distance from a directrix to draw a parabola and develop the general and standard forms of a parabola.

**Standards:** A.CED.2, A.REI.7, F.TF.8, G.GPE.1, G.GPE.2, G.GPE.4 **Pacing:** 9 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **$X^2$ Plus $Y^2$ Equals Radius²**<br><br>Deriving the Equation for a Circle | G.GPE.1 | 2 | Students derive the standard form for the equation of a circle, with the center point and the radius, using the Pythagorean Theorem. Next, the general form for the equation of a circle is introduced. Students are shown how to rewrite the equation of a circle from general form to standard form by completing the square. In standard form, the radius and center point can be readily identified. | • The standard form of the equation of a circle centered at $(h, k)$ with radius $r$ can be expressed as $(x - h)^2 + (y - k)^2 = r^2$.<br>• The equation for a circle in general form is $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $A$, $C$, $D$, $E$, and $F$ are constants, $A = C$, and $x \neq y$.<br>• An equation written in general form can be written in standard form using the algebraic procedure called completing the square. |
| 2 | **A Blip on the Radar**<br><br>Determining Points On a Circle | A.REI.7<br>G.GPE.1<br>G.GPE.4 | 1 | Students use the Pythagorean Theorem and the Distance Formula to determine whether a point lies on a circle, given the location of the center point (which may or may not be at the origin) and either the coordinates of a point on the circle or the circle's radius or diameter. | • The Pythagorean Theorem can be used to determine whether a point lies on the circumference of a circle when the center point is located at the origin and the length of the radius is given.<br>• The Pythagorean Theorem, the Distance Formula, and symmetry can be used to determine whether a point lies on the circumference of a circle when the center point is not located at the origin and the coordinates of a point on the circumference of a circle are given.<br>• The coordinates of the points at which a circle and line intersect can be determined algebraically by writing equations for the line and for the circle, substituting the expression representing the $y$-value of the line into the equation of the circle, and then solving the quadratic equation.<br>• Segments drawn tangent to the same circle from the same exterior point are congruent.<br>• The equation of a line drawn tangent to a circle can be determined given the center point of the circle, a radius drawn to the point of tangency, and the coordinates of the point of tangency. |
| 3 | **$\sin^2 θ$ Plus $\cos^2 θ$ Equals 1²**<br><br>The Pythagorean Identity | F.TF.8 | 2 | Students investigate the sine, cosine, and tangent of angle measures that form right triangles in Quadrants II, III, and IV. They use a unit circle to determine the signs of each trigonometric ratio in each quadrant of the coordinate plane. Students then prove the Pythagorean Identity, $\sin^2 θ + \cos^2 θ = 1^2$, and write this identity in different forms. Students use the Pythagorean Identity to determine the sine, cosine, and tangent of angle measures in given quadrants. Students summarize by labeling different angle measures on the unit circle with degrees and radians, identifying the sign of the sine, cosine, and tangent of each angle measure, and then calculating each trigonometric ratio. | • The Pythagorean identity states that $\sin^2 θ + \cos^2 θ = 1$, where θ represents an angle measure.<br>• The ratios sine, cosine, and tangent may be positive or negative, depending on which quadrant of the coordinate plane the reference angle and reference triangle are drawn.<br>• The sines and cosines of angle measures in different quadrants of the coordinate plane are related by symmetry. |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 4 | **Going the Equidistance**<br>Equation of a Parabola | G.GPE.2 | 2 | The focus and directrix of a parabola are introduced through an exploratory activity. Next, students use the focus and directrix to write the equation of a parabola in both general and standard form. They derive the standard form of a parabola algebraically to make sense of the constant $p$ in the equation and use this constant to graph parabolas. Students then use the Distance Formula to determine the equation for the set of points equidistant from the focus and the directrix when the vertex is not at the origin. Finally, students apply characteristics of parabolas—including the axis of symmetry, vertex, and concavity—to solve problems. | • A parabola is the locus of points in a plane that are equidistant from a fixed point (the focus) and a fixed line (the directrix).<br>• The focus and directrix of a parabola can be used to derive the equation of the parabola.<br>• Parabolas can be described by their concavity.<br>• The standard form for the equation of a parabola with vertex at the origin can be written in the form $x^2 = 4py$ (symmetric with respect to the $y$-axis) or $y^2= 4px$ (symmetric with respect to the $x$-axis), where $p$ is the distance from the vertex to the focus.<br>• The standard form for the equation of a parabola with vertex at the origin, $y^2 = 4px$ or $x^2 = 4py$ can be derived using the Distance Formula and the definitions of focus, directrix, and parabola.<br>• In the standard form for the equation of a parabola centered at the origin, $y^2 = 4px$ or $x^2 = 4py$ the value of $p$ is positive when the parabola is concave up or concave right and the value of $p$ is negative when the parabola is concave down or concave left.<br>• The standard forms of parabolas with vertex $(h, k)$ are $(x - h)^2 = 4p(y - k)$ and $(y - k)^2 = 4p(x - h)$.<br>• The characteristics of parabolas can be used to solve real-world problems. |
| **Learning Individually with MATHia or Skills Practice** | | G.GPE.1 | 2 | Students are given a circle on the coordinate plane with a defined center. They use the Pythagorean Theorem to derive the standard form for the equation of a circle. Students then are given an equation for a circle. They rewrite the equation if necessary in standard form to identify the radius and center of the circle. | |

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# 5 Making Informed Decisions

**Pacing: 15 Days**

## Topic 1: Independence and Conditional Probability

Students are introduced to compound probability. They explore various probability models and calculate compound probabilities with independent and dependent events in a variety of problem situations. The emphasis is on modeling and analyzing sample spaces to determine rules for calculating probabilities in different situations.

**Standards:** S.CP.1, S.CP.2, S.CP.7, S.CP.8(+) **Pacing:** 6 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **What Are the Chances?**<br>Compound Sample Spaces | S.CP.1 | 1 | Students learn strategies for determining the sample space of compound events. They begin with a review of terms associated with probability such as sample space, event, and probability model. For each given situation, students list the sample space, construct a probability model, and differentiate between uniform and nonuniform probability models. They examine examples of tree diagrams and then, for different situations, create their own tree diagrams and organized lists of the corresponding sample spaces. They analyze the sample space in each situation, distinguishing between situations that involve independent events from disjoint sets and those that involve dependent events from intersecting sets. Students identify a situation comprising dependent events that does not allow repetitions. The Counting Principle is stated and discussed as a shortcut for determining the size of a sample space. | • The probability of an event is the ratio of the number of desired outcomes to the total number of possible outcomes.<br>• An outcome is a result of an experiment. A sample space is all of the possible outcomes in a probability situation. An event is an outcome or set of outcomes in a sample space.<br>• A probability model lists the possible outcomes and the probability of each outcome. The sum of the probabilities in the model equals one.<br>• The complement of an event is an event that contains all the outcomes in the sample space that are not outcomes in the event.<br>• A non-uniform probability model is a model in which all of the outcomes are not equal.<br>• Disjoint sets do not have common elements. Intersecting sets have at least one common element.<br>• Independent events are events for which the occurrence of one event has no impact on the occurrence of the other event. Dependent events are events for which the occurrence of one event has an impact on the occurrence of the following events.<br>• The Counting Principle states: "If an action *A* can occur in *m* ways and for each of these *m* ways, an action *B* can occur in *n* ways, then actions *A* and *B* can occur in $m \cdot n$ ways." |
| 2 | **And?**<br>Compound Probability with "And" | S.CP.2<br>S.CP.8(+) | 1 | Students determine the probability of two or more independent events and two or more dependent events. The Rule of Compound Probability involving and is stated and is used to compute compound probabilities. Various situations present students with opportunities to construct tree diagrams, create organized lists, and compute the probabilities of compound events. | • A compound event is an event that consists of two or more events.<br>• The Rule of Compound Probability Involving and states: "If Event *A* and Event *B* are independent, then the probability that Event *A* happens and Event *B* happens is the product of the probability that Event *A* happens and the probability that Event *B* happens, given that Event *A* has happened." However *A* happening has no influence on *B* happening. Using probability notation, the Rule of Compound Probability involving and is $P(A \text{ and } B) = P(A) \cdot P(B)$.<br>• The Rule of Compound Probability Involving and states: "If Event *A* and Event *B* are dependent, then the probability that Event *A* happens and Event *B* happens is the product of the probability that Event *A* happens and the probability that Event *B* happens, given that Event *A* has happened." Using probability notation, the Rule of Compound Probability involving and is $P(A \text{ and } B) = P(A) \cdot P(B\|A)$. |

*Pacing listed in 45-minute days
† The full intent of the standard is not met in this lesson. Students make conjectures about these theorems; they will prove them and fully meet the standard in future lessons.

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **Or?**<br><br>Compound Probability with "Or" | S.CP.7 | 1 | Students determine the probability of one or another independent events and the probability of one or another dependent events. The Addition Rule for Probability is stated and used to compute probabilities. Several situations present students with the opportunity to construct tree diagrams, create organized lists, complete tables, and compute $P(A)$, $P(B)$, $P(A \text{ and } B)$, and $P(A \text{ or } B)$ with respect to the problem situation. Students create a graphic organizer to record the different types of compound events they have studied; independent events $P(A \text{ and } B)$, independent events $P(A \text{ or } B)$, dependent events $P(A \text{ and } B)$, dependent events $P(A \text{ or } B)$. | • A compound event is an event that consists of two or more events.<br>• The Addition Rule for Probability states: "The probability that Event *A* occurs or Event *B* occurs is the probability that Event *A* occurs plus the probability that Event *B* minus the probability that both *A* and *B* occur." Using probability notation, the Addition Rule for Probability is $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$. |
| 4 | **And, Or, and More!**<br><br>Calculating Compound Probability | S.CP.2<br>S.CP.7<br>S.CP.8(+) | 1 | Students analyze scenarios involving a standard deck of playing cards, choosing committee members, and selecting items from a menu and determine compound probabilities. Students determine the probability of independent events $P(A \text{ and } B)$ with replacement, independent events $P(A \text{ or } B)$ with replacement, dependent events $P(A \text{ and } B)$ without replacement, and dependent events $P(A \text{ or } B)$ without replacement. | • Situations "with replacement" generally involve independent events. Whether or not the first event happens has no effect on the second event.<br>• Situations "without replacement" generally involve dependent events. If the first event occurs, it has an impact on the probability of subsequent events. |
| **Learning Individually with MATHia or Skills Practice** | | S.CP.2<br>S.CP.3<br>S.CP.6 | 2 | Students define independent events. They investigate different scenarios to determine whether the events given are independent or not independent. Students then investigate compound probability with "and" and use the equation $P(A \text{ and } B) = P(A)P(B)$ to verify whether two events are independent or not. They use an interactive Explore Tool to explore probability using area and random points. Students then explore the idea of conditional probability, using the interactive tool to visualize the conditional probability formula $P(A \mid B)$ = <stack>$P(A \text{ and } B) / P(B)$</stack>. Finally, they apply what they know about conditional probability to make predictions and check for the independence of events using the Explore Tool. | |

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## Topic 2: Computing Probabilities

Students encounter more compound probability concepts and counting strategies. Compound probability concepts are presented using two-way frequency tables, conditional probability, and independent trials. The counting strategies include permutations, permutations with repetition, circular permutations, and combinations. The last lesson focuses on geometric probability and expected value.

**Standards:** S.CP.3, S.CP.4, S.CP.5, S.CP.6, S.CP.9(+), S.MD.6, S.MD.7 **Pacing:** 9 Days

| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 1 | **Table Talk**<br><br>Compound Probability for Data Displayed in Two-Way Tables | S.CP.4 | 1 | Two number cubes and the results of surveys are the contexts for creating sample spaces, organized lists, and tables. The converse of the multiplication rule is stated and used to determine when events are independent. The terms *frequency*, *frequency table*, *two-way frequency table*, *relative frequency*, and *two-way relative frequency table* are introduced. Students complete these types of tables and use the tables to answer questions related to the situations. Students convert ratios to percents. | • A two-way table is a table that shows the relationship between two data sets, one organized in rows and one organized in columns.<br>• A frequency table is a table that shows the frequency of an item, number, or event appearing in a sample space.<br>• A two-way frequency table or contingency table shows the number of data points and their frequencies for two variables.<br>• A relative frequency is the ratio of occurrences within a category to the total number of occurrences.<br>• A two-way relative frequency table displays the relative frequencies for two categories of data.<br>• Two-way tables can be used to determine the probabilities of compound events.<br>• The converse of the multiplication rule for probability states: "If the probability of two events $A$ and $B$ occurring together is $P(A) \cdot P(B)$, then the two events are independent." |
| 2 | **It All Depends**<br><br>Conditional Probability | S.CP.3<br>S.CP.5<br>S.CP.6 | 1 | Rolling two number cubes and calculating the sum is once again used to generate a two-way data table listing the possible outcomes. Different events are described and students calculate $P(A)$, $P(B)$, and $P(A \text{ and } B)$. The term *conditional probability*, $P(B\|A)$ is defined. Students derive a formula for computing conditional probability, $P(B\|A) = (P(A \text{ and } B))/P(A)$. The conditional probability formula is applied to several different situations. | • Conditional probability is the probability of Event $B$, given that Event $A$ has already occurred.<br>• The notation for conditional probability is $P(B\|A)$, which reads, "the probability of Event $B$, given Event $A$."<br>• When $P(B\|A) = P(B)$, the two events, $A$ and $B$, are independent.<br>• When $P(B\|A) \neq P(B)$, the two events, $A$ and $B$, are dependent.<br>• The conditional probability formula is stated as $P(B\|A) = P(A \text{ and } B)/P(A)$. |

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| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 3 | **Give Me 5!**<br>Permutations and Combinations | S.CP.9(+) | 2 | The terms *factorial*, *permutation*, and *combination* are defined. Students derive the formulas to calculate permutations and combinations, then apply them in different situations. Situations involve permutations with and without repeated elements. Students answer questions, complete tables, and make the connections necessary to develop additional formulas related to combinations and permutations. Circular permutations are introduced. Students conclude that the formula for the circular permutation of n objects is $(n - 1)!$. | • The factorial of $n$, which is written with an exclamation mark as $n!$, is the product of all non-negative integers less than or equal to $n$: $n(n - 1)(n - 2) \ldots$<br>• A permutation is an ordered arrangement of items without repetition.<br>• The notation denoting a permutation or $r$ elements taken from a collection of $n$ items is: ${}_nP_r = P(n, r) = P_r^n$.<br>• The formula used to compute the number of permutations, $P$, of $r$ elements chosen from $n$ elements is: ${}_nP_r = n!/(n - r)!$.<br>• A combination is an unordered collection of items.<br>• The notation denoting a combination or $r$ elements taken from a collection of $n$ elements is: ${}_nC_r = C(n, r) = C_r^n$.<br>• The formula used to compute the number of combinations, $C$, of $r$ elements chosen from $n$ elements is: ${}_nC_r = n!/(n - r)!r!$.<br>• The formula for the number of permutations of n elements with $k$ copies of an element is $n!/k!$.<br>• The formula for the number of permutations of $n$ elements with $k$ copies of one element and $h$ copies of another element is $n!/(k!h!)$.<br>• The circular permutation of $n$ objects is $(n - 1)!$. |
| 4 | **A Different Kind of Court Trial**<br>Independent Trials | S.CP.9(+) | 1 | Situations in this lesson focus on multiple trials for two independent events. Making free throw shots, rolling number cubes, and rolling a tetrahedron are used to generate the probabilities of two independent events. Outcomes are organized in a table and the table is connected to Pascal's Triangle. Students use Pascal's Triangle to compute the probability of an occurrence. A formula using combinations is applied to different situations to calculate probabilities for two independent events over multiple trials. | • If the probability of Event $A$ is $p$ and the probability of Event $B$ is $1 - p$, then the probability of Event $A$ occurring $r$ times and Event $B$ occurring $n - r$ times in $n$ trials is: $P$($A$ occurring $r$ times and $B$ occurring $n - r$ times) or ${}_nC_r\,(p)^r\,(1 - p)^{n-r}$. |

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| Lesson | Title / Subtitle | Standards | Pacing* | Lesson Summary | Essential Ideas |
|---|---|---|---|---|---|
| 5 | **What Do You Expect?**<br>Expected Value | S.MD.6<br>S.MD.7 | 2 | The terms *geometric probability* and *expected value* are introduced in this lesson. Dartboards containing geometric shapes are used to determine geometric probabilities. Money wheels divided into eight equal regions are used to determine expected values. | • Geometric probability is the likelihood of an event occurring based on geometric relationships such as area, surface area, volume, and so on.<br>• Expected value is the sum of the values of a random variable with each value multiplied by its probability of occurrence. |
| **Learning Individually with MATHia or Skills Practice** | | S.CP.4<br>S.CP.5<br>S.CP.7 | 2 | Students review how to read a two-way frequency table and construct a relative frequency table. They then use two-way frequency tables to determine probabilities, including conditional and other compound probabilities; they use information from frequency tables to check for the independence of events. Students apply the concept of conditional probability in a variety of different situations involving a change in the sample space as a result of an event. Students determine probabilities of compound events from two-way frequency tables via the Addition Rule. | |

**Total Days: 168**

**Learning Together: 107**
**Learning Individually: 61**

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